Over View of the course

This course is designed for undergraduate program students who have been taken the static part of the course in early course. The course has five main Chapters arranged according to the difficulty of learning out comes. The first Chapter introduces the Maxwell’s equation with their applications in different physical problems. The second Chapter summarizes the conservation laws in electrodynamics. The third Chapter focuses on formulation of potentials for different condition; when we have different sources. The fourth Chapter describes the phenomena of electromagnetic radiation. The causes and applications of electromagnetic radiations were being presented in this chapter. The last Chapter, chapter five discusses the relativistic electrodynamics; relativistic fields and potentials, tensor representation of fields and potentials are also the concern of the chapter.

Consequently, at the end of the course the following minimum competency are expected:

- apply Maxwell’s equation to variety of physical systems,
- describe electromagnetic phenomena with the aid of potentials,
- demonstrate understanding how electric potential and fields transform,
- solve problems applying potential formalism and understand that the results are independent of the approaches one used,
- demonstrate understanding of the process of electromagnetic radiation,
- Relate electrodynamics with relativity.
3. Potential Formulation

3.1 Scalar and vector potentials

In this chapter we try to find the general solutions of Maxwell’s equations.

\[
\begin{align*}
(i) \quad \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
(ii) \quad \nabla \cdot \vec{B} &= 0 \\
(iii) \quad \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
(iv) \quad \nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{align*}
\]

(3.1)

In the static field cases the general solution of \(\vec{E}\) and \(\vec{B}\) for a given \(\rho(\vec{r})\) and \(\vec{J}(\vec{r})\) are given by **Coulomb’s law** and **Biot-Savart law**, respectively. Then, now for non-static field \(\vec{E}\) due to \(\rho(\vec{r}, t)\) and \(\vec{B}\) due to \(\vec{J}(\vec{r}, t)\) the result is different and we try to generalize the Coulomb’s law and the Biot-Savart law respectively

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{r^2} d\tau' \hat{r} \quad \text{(Coulomb’s law)}
\]

\[
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{r^2} d\tau' \times \hat{r} \quad \text{(Biot-Savart law)}
\]

In electrostatic \(\nabla \times \vec{E} = 0\) and \(\vec{E} = -\nabla V\). In electrodynamics this is no longer possible, because the curl of \(\vec{E}\) is non-zero. But \(\vec{B}\) is divergence less and we write

\[
\vec{B} = \nabla \times \vec{A} \quad \text{(3.2)}
\]

as in magneto statics. Putting this in to Faraday’s law Eq. (iii) of (3.1) gives
This means that we have a quantity whose curl does vanish; if this is the case, we can write it as the gradient of a scalar, which is

\[ \nabla \times [\vec{E} + \frac{\partial}{\partial t} \vec{A}] = -\nabla V \]

or

\[ \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \]  \hspace{1cm} (3.3)

Now we substitute equations (3.2) and (3.3) in to (ii) and (iii) to check whether they satisfy the homogeneous Maxwell’s equations.

\[ \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0 \]

Divergence of any curl of vector is zero.

\[ \nabla \times \vec{E} = \nabla \times (-\nabla V - \frac{\partial \vec{A}}{\partial t}) = -\nabla \times (\nabla V) - \frac{\partial}{\partial t} (\nabla \times \vec{A}) = 0 - \frac{\partial}{\partial t} \vec{B}. \]

Since \[ \nabla \times (\nabla V) = 0 \]

Furthermore, we substitute (3.3) in to (i), to see the form of equations in the non-homogeneous part of electric field. That is

\[ \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \Rightarrow \nabla \cdot (-\nabla V - \frac{\partial \vec{A}}{\partial t}) = \frac{1}{\varepsilon_0} \rho \]

\[ \nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{1}{\varepsilon_0} \rho \] \hspace{1cm} (3.4)

Similarly, we insert the result obtained in Eqs. (3.2) and (3.3) in to (iv) to see the form of Ampere-s-Maxwell’s equation in inhomogeneous part.
Rearranging terms gives

\[
\left( \nabla^2 \vec{A} - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{A} \right) - \nabla (\nabla \cdot \vec{A} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} V) = -\mu_0 \vec{J} \quad (3.5)
\]

Equations (3.4) and (3.5) contain all the information in Maxwell’s equations.

Example 1: For the configuration

\[
V = 0, \quad A = \begin{cases} 
\frac{\mu_0 k}{4c} (ct) \left( \frac{x}{l} \right)^2 \hat{z}, & \text{for } |x| < ct, \\
0, & \text{for } |x| > ct.
\end{cases}
\]

Where \( k \) is a constant, and \( C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \). Consider a rectangular box of length \( l \), width \( w \), and height \( h \), situated a distance \( d \) above the \( yz \) plane Fig. below.

a) Find the energy in the box at time \( t_1 = \frac{d}{c} \) and \( t_2 = \left( \frac{h+d}{c} \right) \).

b) Find the Poynting vector, and determine the energy per unit time flowing into the box during the interval \( t_1 < t < t_2 \).

c) Integrate the result in (b) from \( t_1 < t < t_2 \) and confirm that the increase in energy (part (a)) equals the net influx.

Solution: First determine the electric and magnetic fields using equations (3.2) and (3.3):
Electrodynamics II (PHYS 3082) Lecture Notes By: Kefale M.

\[ \vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 k}{2} (ct-|x|) \hat{z}, \]
\[ \vec{B} = \nabla \times \vec{A} = -\frac{\mu_0 k}{4c} \frac{\partial}{\partial x} (ct-|x|)^2 \hat{y} = \pm \frac{\mu_0 k}{2c} (ct-|x|) \hat{y}, \]

Plus for \( x > 0 \) and minus for \( < 0 \). Now we can calculate the required as follow

\[ W = \frac{1}{2} \int \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau. \]

At \( t_1 = d/c \), \( x \geq d = ct_1 \), so \( \vec{E} = 0 \), \( \vec{B} = 0 \) and hence \( W(t_1) = 0 \).

At \( t_2 = (d+h)/c \), \( ct_2 = d+h \):

\[ \vec{E} = -\frac{\mu_0 k}{2} (d+h-x) \hat{z}, \quad \vec{B} = \frac{1}{c} \frac{\mu_0 k}{2} (d+h-x) \hat{y}, \]

so \( B^2 = \frac{1}{c^2} E^2 \), and \( \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) = \varepsilon_0 \left( E^2 + \frac{1}{\mu_0 \varepsilon_0} \frac{1}{c^2} E^2 \right) = 2\varepsilon_0 E^2. \)

Therefore,

\[ W(t_2) = \frac{1}{2} \left( 2\varepsilon_0 \right) \frac{\mu_0 k^2}{4} \int_d^{d+h} (d+h-x)^2 dx (\omega) = \frac{(\varepsilon_0 \mu_0 k^2 \hbar \omega)}{4} \left[ -\left( \frac{(d+h-x)}{3} \right) \right]_d^{d+h} \]

\[ = \frac{(\varepsilon_0 \mu_0^2 k^2 \hbar \omega^3)}{12}. \]

b) \( \vec{S}(x) = \frac{1}{\mu_0} (\vec{B} \times \vec{E}) = \frac{1}{\mu_0 c} E^2 [-\hat{z} \times (\pm \hat{y})] = \pm \frac{1}{\mu_0 c} E^2 \hat{x} = \pm \frac{\mu_0 k^2}{4c} (ct-|x|)^2 \hat{x} \)

(Plus is sign for \( x > 0 \)). For \( |x| > ct \), \( \vec{S} = 0 \).

So the energy per unit time entering the box in this time interval is

\[ \frac{dW}{dt} = P = \int \vec{S}(d) \cdot d\vec{a} = \pm \frac{\mu_0 \alpha^2 \hbar \omega}{4c} (ct-d)^2. \]
c) \( W = \int_{0}^{t} \mathcal{P} dt = \frac{\mu_0 k^2 \hbar}{4c} (ct)^{d-k}/c \frac{d}{dc} \left( \frac{d}{dc} + \frac{d}{dc} \right) \) 

\[
\frac{d}{dc}\left( \frac{d}{dc} + \frac{d}{dc} \right) = \frac{\mu_0 k^2 \hbar n^3}{12c^2}.
\]

**Activity 1:** Show that the differential equations for \( V \) and \( \vec{A} \) in (3.4) and (3.5) can be written in the more systematic form.

\[
\begin{align*}
\square^2 V + \frac{\partial L}{\partial t} &= \frac{1}{\varepsilon_0} \rho, \\
\square^2 \vec{A} \cdot \nabla L &= \mu_0 j, \quad \text{where} \quad \square^2 \equiv \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \quad \text{and} \quad L \equiv \vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}.
\end{align*}
\]

### 3.2 Coulomb’s and Lorentz’s Gauge

#### 3.2.1 Gauge Transformation

For conditions not changing \( \vec{E} \) and \( \vec{B} \) we can impose extra conditions on \( V \) and \( \vec{A} \). Let

\[
\vec{A}' = \vec{A} + \alpha \quad \text{and} \quad v' = v + \beta
\]

The two \( \vec{A}' \)'s (\( \vec{A} \) and \( \vec{A} \)) gives the same \( \vec{B} \). That is

\[
\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \alpha \Rightarrow \vec{\nabla} \times \alpha = \vec{\nabla} \times \vec{A}' - \vec{\nabla} \times \vec{A} = \vec{B}' - \vec{B} = 0
\]

\[
\therefore \vec{\nabla} \times \alpha = 0.
\]

Hence the curl of \( \alpha \) vanish, we can write \( \alpha = \nabla \lambda \)

The two potentials also give the same \( \vec{E} \), so that we can write

\[
\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} - \frac{\partial \alpha}{\partial t}, \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}
\]

\[
\vec{E}' - \vec{E} = \vec{\nabla} V - \vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} - \frac{\partial \alpha}{\partial t} + \vec{\nabla} V + \frac{\partial \vec{A}}{\partial t} = 0
\]

\[
\Rightarrow \vec{\nabla} \beta + \frac{\partial \alpha}{\partial t} = 0
\]

We may write this last equation as:

\[
\vec{\nabla} \beta + \frac{\partial \lambda}{\partial t} = 0 \Rightarrow \vec{\nabla} \left( \beta + \frac{\partial \lambda}{\partial t} \right) = 0,
\]
Where $\lambda$ depends on time but not on position. Therefore, the change we made in $V$ and $\vec{A}$ that can be generalized:

$$
\begin{align*}
\vec{A}' &= \vec{A} + \nabla \lambda, \\
V' &= V - \frac{\partial \lambda}{\partial t} 
\end{align*}
$$

(3.7)

### Electrodynamics II: Lecture Two

#### 3.2.2 The Coulomb Gauge

In magneto statics

$$
\vec{\nabla} \cdot \vec{A} = 0
$$

(3.8)

Inserting this into (3.4) we obtain

$$
\nabla^2 V = -\frac{1}{\varepsilon_0} \rho
$$

(3.9)

This is Poisson’s equation and for $V = 0$ at infinity, the solution reads

$$
V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}', t)}{r^2} d\tau
$$

(3.10)

The advantage of Coulomb gauge is that the scalar potential is simple to calculate; the disadvantage is that $\vec{A}$ is particularly difficult to calculate. In Coulomb gauge Eq. (3.5) can be written as:

$$
\nabla^2 \vec{A} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \varepsilon_0 \mu_0 \vec{\nabla} \left( \frac{\partial V}{\partial t} \right)
$$

(3.11)

#### 3.2.3 The Lorentz gauge

Equations (3.6) and (3.7) would be much simplified if we could demand that the potentials satisfy the supplementary condition

$$
\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0
$$

(3.12)
This is known as the **Lorenz gauge** condition and it is aimed to eliminate the middle term in (3.5).

\[ \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}. \quad (3.13) \]

And Eq. (3.4) becomes

\[ \nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}. \quad (3.14) \]

Lorentz gauge treat \(V\) and \(\vec{A}\) on an equal footing: the same differential operator

\[ \Box = \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \quad (3.15) \]

This equation is called the **d’Alembertian**, occurs in both equations (i.e. 13 and 14)

\[
i) \quad \Box V = -\frac{1}{\varepsilon_0} \rho, \\
ii) \quad \Box \vec{A} = -\mu_0 \vec{J}. \quad (3.16)
\]

In the Lorentz gauge \(V\) and \(\vec{A}\) satisfies the inhomogeneous wave equation, with a source term (in place of zero) on the right hand side. With the transformation performed using the above gauge transformation, the whole electrodynamics reduces to the problem of solving the inhomogeneous wave equation for specified sources.

### 3.3 Continuous charge Distribution

#### 3.3.1 Retarded Potentials

In **electrodynamics**, the **retarded potentials** are the **electromagnetic potentials** for the **electromagnetic field** generated by **time-varying electric current** or **charge distributions** in the past. The fields propagate at the **speed of light** \(c\), so the delay of the fields connecting **cause and effect** at earlier and later times is an important factor.

**Note to open the link:** Right click, then open hyperlink

In the static case Eq. (3.16) reduce to four copies of Poisson’s equation
With the familiar solutions

\[ V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\vec{r}')}{\varepsilon} d\tau \]
\[ \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \left( \int \frac{\vec{J}(\vec{r}')}{\varepsilon} d\tau \right) \]

Where \( \tau \) as always, is the distance from the source point \( r' \) to the field point \( r \) as shown in Fig. below.

Electromagnetic information (news) travels at the speed of light. In the non-static case, it is not the status of the source that light matters, but rather its condition at some earlier time \( r',t \) (called the retarded time) when the message left. Since this message must travel a distance \( \tau \), the delay is \( \tau/c \) where \( c \) is the speed of light.

\[ t_r = t - \tau/c \]  

(3.18)

Based on this fact we can generalize the static sources of (3.17) to non-static sources as

\[ V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\vec{r}',t_r)}{\varepsilon} d\tau \]
\[ \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \left( \int \frac{\vec{J}(\vec{r}',t_r)}{\varepsilon} d\tau \right) \]

Where is \( \rho(\vec{r}',t) \) the charge density that prevailed \( \vec{r}' \) at point at the retarded rime \( r,t \). Because the integrands are evaluated at the retarded time, these are retarded potentials. Now let us check that the retarded potentials obey the Lorentz condition and the potential too. The integrand in Eq. (3.19) depends on:
\[ \mathbf{r} = |\vec{r} - \vec{r}'| \text{ and } t_r = t - \mathbf{r}/c. \]

\[
V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}', t_r)}{r} d\tau
\]

\[
\nabla V = \frac{1}{4\pi\varepsilon_0} \int \left[ (\nabla \rho) \frac{1}{r} + \rho \frac{1}{r^2} \right] d\tau
\]  \hspace{1cm} (3.20)

\[
\nabla \mathbf{\rho} = \frac{\partial \rho}{\partial t_r} \frac{\partial \mathbf{r}}{\partial t_r} \cdot \hat{\mathbf{e}} = -\mathbf{e} \frac{1}{c} \nabla V
\]  \hspace{1cm} (3.21)

But we know that \( \nabla \mathbf{\mathbf{e}} = \mathbf{\hat{e}} \) and \( \nabla (1/\mathbf{\mathbf{e}}) = -\mathbf{\hat{e}} / \mathbf{\mathbf{e}}^2 \)

With this

\[
\nabla V = \frac{1}{4\pi\varepsilon_0} \int \left[ -\frac{\dot{\rho}}{c} \frac{\mathbf{\hat{e}}}{\mathbf{\mathbf{e}}} - \rho \frac{\mathbf{\hat{e}}}{\mathbf{\mathbf{e}}^2} \right] d\tau'
\]  \hspace{1cm} (3.22)

Taking the divergence of Eq. (3.22)

\[
\nabla^2 V = \frac{1}{4\pi\varepsilon_0} \int \left[ -\frac{1}{c^2} \left( \frac{\ddot{\mathbf{e}}}{\mathbf{\mathbf{e}}} \right) \nabla \rho + \rho \nabla \frac{1}{\mathbf{\mathbf{e}}} \cdot \left( \frac{\mathbf{\hat{e}}}{\mathbf{\mathbf{e}}^2} \right) \right] d\tau'
\]

\[
\nabla \rho = \frac{\partial}{\partial t_r} \rho \frac{\partial \mathbf{r}}{\partial t_r} \cdot \hat{\mathbf{e}} = \frac{1}{c} \rho \nabla \mathbf{\mathbf{e}} = -\frac{1}{c} \ddot{\mathbf{e}} \quad \text{and} \quad \nabla \cdot \left( \frac{\mathbf{\hat{e}}}{\mathbf{\mathbf{e}}^2} \right) = -\frac{1}{\mathbf{\mathbf{e}}^2}
\]

\[
\nabla \left( \frac{\mathbf{\hat{e}}}{\mathbf{\mathbf{e}}^2} \right) = 4\pi\delta^3(\mathbf{\mathbf{e}})
\]

\[
\nabla^2 V = \frac{1}{4\pi\varepsilon_0} \int \left[ \frac{1}{c^2} \ddot{\rho} - 4\pi\delta^3(\mathbf{\mathbf{e}}) \right] d\tau'
\]

Hence

\[
\nabla^2 V = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} V - \frac{1}{\varepsilon_0} \rho(\mathbf{\mathbf{e}}, t)
\]

Or

\[
\nabla^2 V - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} V = -\frac{1}{\varepsilon_0} \rho(\mathbf{\mathbf{e}}, t)
\]
Therefore, the retarded potential (3.19) satisfies the inhomogeneous wave equation (3.16).

Similarly, we can show that the solution in vector potential \( \vec{A} \) of Eq. (3.19) satisfies the Lorentz gauge condition.

First let show that \( \vec{V} \cdot (\vec{J}/\tau) = \frac{1}{\tau} (\vec{V} \cdot \vec{J}) + \frac{1}{\tau} (\vec{V}' \cdot \vec{J}) - \vec{V}'(\vec{J}/\tau) \).

Here, \( \vec{V} \) represent derivatives respect to \( \vec{r} \) and \( \vec{V}' \) represent derivatives respect to \( \vec{r}' \).

\[
\vec{V} \cdot (\vec{J}/\tau) = \frac{1}{\tau} (\vec{V} \cdot \vec{J}) + \frac{1}{\tau} (\vec{V}(1/\tau)),
\]

\[
\nabla' (\vec{J}/\tau) = \frac{1}{\tau} (\nabla' \cdot \vec{J}) + \frac{1}{\tau} (\nabla'(1/\tau)),
\]

\[
\vec{V}(1/\tau) = \hat{r} \frac{\partial}{\partial r} \left( \frac{1}{|\vec{r}|} \right) = \frac{\hat{r}}{\tau^2}
\]

\[
\nabla'(1/\tau) = \hat{r}' \frac{\partial}{\partial r'} \left( \frac{1}{|\vec{r}'|} \right) = \frac{\hat{r}'}{\tau^2}
\]

\[
\Rightarrow \nabla (1/\tau) = -\nabla' (1/\tau)
\]

\[
\vec{V} \cdot (\vec{J}/\tau) = \frac{1}{\tau} (\vec{V} \cdot \vec{J}) - \vec{J} \cdot (\vec{V}'(1/\tau)) = \frac{1}{\tau} (\vec{V} \cdot \vec{J}) + \frac{1}{\tau} (\vec{V} \cdot \vec{J}) - \vec{V}'(\vec{J}/\tau)
\]

Basically, \( \vec{V} \cdot \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \frac{\partial J_x}{\partial t_r} \cdot \frac{\partial t_r}{\partial x} + \frac{\partial J_y}{\partial t_r} \cdot \frac{\partial t_r}{\partial y} + \frac{\partial J_z}{\partial t_r} \cdot \frac{\partial t_r}{\partial z} \).

\[
\frac{\partial t_r}{\partial x} = -\frac{1}{c} \frac{\partial \tau}{\partial x}, \quad \frac{\partial t_r}{\partial y} = -\frac{1}{c} \frac{\partial \tau}{\partial y}, \quad \frac{\partial t_r}{\partial z} = -\frac{1}{c} \frac{\partial \tau}{\partial z},
\]

And hence

\[
(\vec{V} \cdot \vec{J}) = -\frac{1}{c} \left[ \frac{\partial J_x}{\partial t_r} \frac{\partial \tau}{\partial x} + \frac{\partial J_y}{\partial t_r} \frac{\partial \tau}{\partial y} + \frac{\partial J_z}{\partial t_r} \frac{\partial \tau}{\partial z} \right]
\]

\[
= -\frac{1}{c} \frac{\partial J}{\partial t_r} \cdot (\vec{V} \tau) = -\frac{1}{c} \dot{J}(\vec{V} \tau)
\]
Similarly,

\[
(\vec{\nabla} \cdot \vec{J}) = -\frac{1}{c} \frac{\partial J}{\partial t},
\]

\[
= -\frac{1}{c} \rho \cdot \vec{v} - \frac{\partial \rho}{\partial t}.
\]

where \(\vec{\nabla} \cdot \vec{J}(\vec{r}, t) = -\frac{\partial \rho}{\partial t}\) from continuity. Moreover, \(\vec{\nabla} \cdot \vec{J}(\vec{r}, t, \epsilon) = t - \epsilon/c\) depends on \(\vec{r}\) explicitly and through \(\epsilon\), where as it depends on \(\vec{r}\) only through \(\epsilon\).

With these relation,

\[
\vec{\nabla} \cdot (\vec{J}/\epsilon) = \frac{1}{\epsilon} \left[ -\frac{\partial J}{\partial t} \cdot (\vec{\nabla} \epsilon) \right] + \frac{1}{\epsilon} \left[ -\frac{\partial \rho}{\partial t} \cdot \vec{v} + \frac{1}{c} \frac{\partial J}{\partial t} \cdot (\vec{\nabla} \epsilon) \right] - \vec{\nabla} \cdot (\vec{J}/\epsilon)
\]

\[
= -\frac{1}{c \epsilon} \left[ -\frac{\partial J}{\partial t} \cdot (\vec{\nabla} \epsilon) \right] - \frac{1}{\epsilon} \frac{\partial \rho}{\partial t} - \frac{1}{c \epsilon} \frac{\partial J}{\partial t} \cdot (\vec{\nabla} \epsilon) - \vec{\nabla} \cdot (\vec{J}/\epsilon).
\]

Using \((\vec{\nabla} \epsilon) = - (\vec{\nabla} \epsilon)\) we obtain

\[
\vec{\nabla} \cdot (\vec{J}/\epsilon) = -\frac{1}{\epsilon} \frac{\partial \rho}{\partial t} - \vec{\nabla} \cdot (\vec{J}/\epsilon).
\]

Now we can apply the divergence on \(\vec{A}\)

\[
\vec{\nabla} \cdot \vec{A} = \frac{\mu_0}{4\pi} \oint \vec{A} \cdot d\vec{a} = \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{a} = \frac{\mu_0}{4\pi} \oint (\vec{J}/\epsilon) \cdot d\vec{a}
\]

\[
= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left[ \frac{1}{4\pi \epsilon_0} \int \vec{E} \cdot d\vec{a} \right] \frac{\mu_0}{4\pi} \oint (\vec{J}/\epsilon) \cdot d\vec{a}.
\]

The last integral over the surface at \(\epsilon\), where \(\vec{J} = 0\) is
This shows that the Lorentz gauges work for vector potential $\vec{A}$

**Example 2:** A piece of wire bent into a loop, as shown in Fig. below, carries a current that increases linearly with time:

$I(t) = kt$

Calculate the retarded vector potential $A$ at the center. Find the electric field at the center.

Why does this (neutral) wire produce an electric field? (Why can’t you determine the magnetic field from this expression for $A$?)

**Solution:**

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I(t)}{r} d\vec{l} = \frac{\mu_0 k}{4\pi} \int \frac{(t - \tau/\alpha)}{r} d\vec{l} = \frac{\mu_0 k}{4\pi} \left( t \int \frac{d\vec{l}}{r} - \frac{1}{c} \int d\vec{l} \right).$$

For a complete loop, the integral over a closed path is zero, or the last term has no contribution to the vector potential in this case ($\int dl = 0$ loop complete for)

Hence, $\vec{A} = \frac{\mu_0 k t}{4\pi} \left\{ \frac{1}{a} \int d\vec{l} + \frac{1}{b} \left[ d\vec{l} + 2\hat{x} \int_{a}^{b} \frac{dx}{x} \right] \right\}.$
The first integral gives $\pi a \hat{r}$ inner circle, the second integral gives $\pi b \hat{r}$, outer circle and the third integral gives $\ln(b/a)$. Summing up these results

$$\vec{A} = \frac{\mu_0 k t}{2\pi} \ln(b/a) \hat{r}.$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 k}{2\pi} \ln(b/a) \hat{r}.$$

The changing magnetic field induces the electric field. As long as we only know $A$ at one point (the center), we can’t compute $\nabla \times \vec{A} = \vec{B}$

**Activity**

Lorenz and Coulomb gauge-fixing conditions. What is physical difference between these two gauge-fixing conditions? Mathematical expressions are clear but how to we choose one of these means, what they really mean.

**3.3.2 Jefimenko’s equations**

(Dear student please recognized that the topic was consecutive with the previews lesson example equation number…)

In electromagnetism, Jefimenko's equations (named after Oleg D. Jefimenko) describe the behavior of the electric and magnetic fields in terms of the charge and current distributions at retarded times.

Jefimenko's equations are the solution of Maxwell's equations for an assigned distribution of electric charges and currents, under the assumption that there is no electromagnetic field other than the one produced by those charges and currents, that is no electromagnetic field coming from the infinite past.

To derive Jefimenko’s equations we start with the retarded potentials:
In principle, it is straightforward matter to determine the fields:

\[
\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A} \tag{3.24}
\]

The integrands depend on \( \vec{r} \) both explicitly, through \( \varepsilon = |\vec{r} - \vec{r}'| \) in the denominator, and implicitly, through the retarded time \( t_r = t - \varepsilon / c \) in the argument of the numerator.

In the previous section we have calculated \( \nabla V \) and obtained it as presented by Eq. (3.22). That is

\[
\nabla V = \frac{1}{4\pi \varepsilon_0} \int \left[ \frac{\dot{\varrho}}{c} \frac{\hat{\varepsilon}}{\varepsilon} - \frac{\rho}{\varepsilon^2} \right] d\tau' \tag{3.22}
\]

Now we can calculate \( \frac{d\vec{A}}{dt} \)

\[
\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\partial}{\partial t} \frac{\vec{J} (\vec{r}', t_r)}{\varepsilon} d\tau' = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{\varepsilon} d\tau \tag{3.25}
\]

Inserting Eqs. (3.22) and (3.25) into the electric field expression of Eq. (3.24), we obtain

\[
\vec{E} = \frac{1}{4\pi \varepsilon_0} \int \left[ \frac{\dot{\varrho}}{c} \frac{\hat{\varepsilon}}{\varepsilon} + \frac{\dot{\varepsilon}}{\varepsilon^2} \frac{\hat{\varepsilon}}{\varepsilon} - \frac{1}{c^2} \frac{\varepsilon}{\varepsilon^2} \frac{\vec{J}}{\varepsilon^2} \right] d\tau' , \tag{3.26}
\]

Where we used \( \mu_0 = \frac{1}{c^2 \varepsilon_0} \)

Equation (3.26) is the time-dependent generalization of Coulomb’s law, to which it reduces in the static case. In similar way we can calculate the curl of \( \vec{A} \) as
This can be obtained using the property of vector identity

\[ \nabla \times (f \vec{A}) = f (\nabla \times \vec{A}) - \vec{A} \times \nabla f \]

Here the curl of vector \( \vec{J} \) along the x-direction is given by:

\[
(\nabla \times \vec{J})_x = \frac{\partial J_y}{\partial y} - \frac{\partial J_z}{\partial z} \quad \text{and} \quad \frac{\partial J_y}{\partial y} = \frac{\partial J_z}{\partial y} \frac{\partial t_r}{\partial y} = -\frac{1}{c} J_z \frac{\partial \hat{e}}{\partial y} \\

\Rightarrow (\nabla \times \vec{J})_x = -\frac{1}{c} [J_z \frac{\partial \hat{e}}{\partial y} - J_y \frac{\partial \hat{e}}{\partial z}] = \frac{1}{c} [\hat{J} \times \nabla \hat{e}] \\
(\nabla \times \vec{J})_x = \frac{1}{c} [\hat{J} \times \hat{e}], \quad (3.27)
\]

where \( \nabla \hat{e} = \hat{e} \). We have shown that \( \nabla (1/\hat{e}) = -\hat{e} / \hat{e}^2 \), therefore,

\[
\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \left[ \frac{\vec{J}(\vec{r}', t_r)}{\hat{e}^2} + \frac{\dot{\vec{J}}(\vec{r}', t_r)}{c\hat{e}} \right] \times \hat{e} \, dt' 
\]

Equation (3.28) is the time-dependent generalization of the Biot-Sarvart law, to which it reduces in the static case. Equations (3.26) and (3.28) are Jefimenko’s equation.

Example 3:

Suppose the current density changes slowly enough that we can (to good approximation) ignore all higher derivatives in the Taylor expansion

\[ \vec{J}(t_r) = \dot{\vec{J}} (t) + (t_r - t) \ddot{\vec{J}} (t) + \cdots \]

For clarity, in this case the \( r \) dependence is suppressed, which is not at issue). Show that a fortuitous cancelation in Eq. (3.28) yields
This is the Biot-Savart law holds, with J evaluated at the non-retarded time. This means that the quasistatic approximation is actually much better than we had any right to expect: the two errors involved (neglecting retardation and dropping the second term in (3.28) cancel one another, to the first order.

**Solution:** In this approximation we are dropping the higher derivatives of J, so \( \dot{J}(t_r) = \dot{J}(t) \)

\[
\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}',t) \times \hat{\vec{e}}}{\epsilon^2} \, dt' .
\]

But we recall that \( t_r - t = -\epsilon/c \), so that the last equation for B reduced to the first term only.

That is \( \vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}',t) \times \hat{\vec{e}}}{\epsilon^2} \, dt' .\)
Chapter 4: Radiations

In physics, **radiation** is the emission or transmission of energy in the form of waves or particles through space or through a material medium. This includes:

- electromagnetic radiation, such as radio waves, microwaves, infrared, visible light, ultraviolet, x-rays, and gamma radiation (γ)
- particle radiation, such as alpha radiation (α), beta radiation (β), and neutron radiation (particles of non-zero rest energy)
- acoustic radiation, such as ultrasound, sound, and seismic waves (dependent on a physical transmission medium)
- Gravitational radiation, radiation that takes the form of gravitational waves, or ripples in the curvature of space-time.

Radiation is often categorized as either ionizing or non-ionizing depending on the energy of the radiated particles. Ionizing radiation carries more than 10 eV, which is enough to ionize atoms and molecules and break chemical bonds. This is an important distinction due to the large difference in harmfulness to living organisms. A common source of ionizing radiation is radioactive materials that emit α, β, or γ radiation, consisting of helium nuclei, electrons or positrons, and photons, respectively. Other sources include X-rays from medical radiography examinations and muons, mesons, positrons, neutrons and other particles that constitute the secondary cosmic rays that are produced after primary cosmic rays interact with Earth's atmosphere.

### 4.1 Dipole radiation

#### 4.1.1 What is radiation?

Radiation is the irreversible flow of energy away from the source. The second term in (3.53) is the component that carries energy away to infinity (both E and B go as 1/r and hence the Poynting vector is nonzero even at infinity); this is the radiation field and exists only if \( \hat{a} \leq 0 \) Note that these fields are transverse.
Let consider a localized source at the origin of a spherical shell as shown in Fig. 4.1.

The total power passing out through the surface of sphere is the integral of the pointing vector.

$$P(r) = \oint S \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}$$  \hspace{1cm} (4.1)

The power radiated is the limit of this quantity as $r$ goes to infinity.

$$P_{\text{rad}} = \lim_{r \to \infty} P(r)$$  \hspace{1cm} (4.2)

This is the energy (per unit time) that is transported out to infinity and never come back.

The area of the sphere is $4\pi r^2$ in order to radiation exist the Poynting vector cannot decrease at large distance faster than $\frac{1}{r^2}$ (if it behaves like $\frac{1}{r^3}$ then) $p(r)$ would go like $\frac{1}{r}$ and $P_{\text{rad}}$ would be zero). According to Coulomb’s law, electrostatic fields fall of like $\frac{1}{r^2}$ (or even faster, when the total charge is zero), and the Biot-Svart law says that magneto static fields go like $\frac{1}{r^2}$ (or faster), which means that $\mathbf{s}, \sim 1/r^4$ for static configurations. So, static sources do not radiate. But Jefimenko’s equations indicate that time-dependent fields include terms (involving $\dot{\rho}$ and $J$) that go like $\frac{1}{r^3}$; it is these terms that are responsible for electromagnetic radiation.

The study of radiation, then, involves picking out the parts of $\mathbf{B}$ and $\mathbf{E}$ that go like $1/r$ at large distances from the source, and taking the limit as $r \to \infty$.

4.1.2 Electric Dipole Radiation
Let consider two tiny metal spheres connected by a fine wire as shown in Fig. 4.2.

At time $t$ the charge on the upper sphere is $q(t)$ and that of lower sphere is $q(-t)$. If we derive the charge from one end to the other with angular frequency $\omega$:

$$q(t) = q_0 \cos(\omega t)$$  \hspace{1cm} (4.3)

The result is an oscillating electric dipole:

$$\bar{p}(t) = p_0 \cos(\omega t) \hat{z},$$  \hspace{1cm} (4.4)

Where $p_0 = qd$ is the maximum value of the dipole moment. The retarded potential of the two charged spheres at a point $P$ is:

$$V(\vec{r},t) = \frac{q_0}{4\pi\varepsilon_0} \left[ \frac{\cos(\omega(t - \frac{r_+}{c})))}{r_+} - \frac{\cos(\omega(t - \frac{r_-}{c}))}{r_-} \right].$$  \hspace{1cm} (4.5)

$r_\pm$ is given by the cosine law $r_\pm = \sqrt{r^2+(d/2)^2} \mp rd \cos \theta$  \hspace{1cm} (4.6)

The physical dipole can be perfect dipole if the separation distance is extremely small:

**Approximation 1**: $d \ll r$, We make the expansion of (4.6) to the first order in $d$. 

20
\[ r_z = r \left(1 + \left(\frac{d}{2r}\right)^2 \mp \left(\frac{d}{r}\right) \cos \theta\right)^{1/2} \]
\[ = r \left[1 + \frac{d}{2r} \cos \theta + \frac{1}{2} \left(-\frac{d}{2r}\right)^2 + \frac{1}{2} \left(-1/2\right)\left(-\frac{d}{r} \cos \theta\right)^2 + \frac{1}{2} \left(-1/2\right)\left(-\frac{d}{2r}\right)^4 + \ldots\right] \]
\[ \approx r \left[1 \mp \frac{d}{2r} \cos \theta\right] \]
\[ r_z \approx r \left(1 \mp \frac{d}{2r} \cos \theta\right) \quad (4.7) \]

And
\[ \frac{1}{r_z} = \frac{1}{\sqrt{r^2 + \left(\frac{d}{2r}\right)^2 \mp rd \cos \theta}} = \frac{1}{r} \left(1 + \left(\frac{d}{2r}\right)^2 \mp \left(\frac{d}{r} \cos \theta\right)^{-1/2}\right) \]
\[ \approx \frac{1}{r} \left(1 \pm \left(\frac{d}{2r}\right) \cos \theta\right) \]
\[ \Rightarrow \frac{1}{r_z} \approx \frac{1}{r} \left(1 \pm \left(\frac{d}{2r}\right) \cos \theta\right) \quad (4.8) \]

Using Equation (4.7) the cosine term in (4.5) can be approximated as
\[ \cos(\omega(t - r_z/c)) \approx \cos[\omega(t - r/c) \pm \frac{\omega d}{2c} \cos \theta] \]
\[ = \cos[\omega(t - r/c)] \cos\left[\frac{\omega d}{2c} \cos \theta\right] + \sin[\omega(t - r/c)] \sin\left[\frac{\omega d}{2c} \cos \theta\right] \]

For perfect dipole we have another approximation; **approximation 2**: \( d \ll c/\omega \)

Because waves of frequency have a wavelength \( \lambda = \frac{2\pi c}{\omega} \), this accounts to the requirement with this approximation the above last expression becomes \( d \ll r \) with this approximation the above last expression becomes:

\[ \cos(\omega(t - r_z/c)) \approx \cos[\omega(t - r/c)] + \sin[\omega(t - r/c)] \frac{\omega d}{2c} \cos \theta \quad (4.9) \]

Inserting equations (4.8) and (4.9) into (4.5), we get
\[ V(r, \theta, t) = \frac{p_x \cos \theta}{4\pi \varepsilon_0 r} \left\{ -\frac{\omega}{c} \sin[\omega(t - r/c)] + \frac{1}{r} \cos(\omega(t - r/c)) \right\} \quad (4.10) \]
In the static limit ($\omega \to 0$)

$$V = \frac{p_0 \cos \theta}{4\pi \varepsilon_0 r^2}.$$ 

We are interested in the fields that survive at large distances from the source, radiation zone.

**Approximation 3:** $r \gg \frac{c}{\omega} (r \gg \lambda)$

In this limit the second term in (4.10) is proportional to $\frac{1}{r^2}$ and goes to zero. Hence,

\[\dot{I}(t) = -\frac{p_0 \cos \theta}{4\pi \varepsilon_0 r} \frac{\omega}{c} \sin[\omega(t - r/c)] \Rightarrow (4.11)\]

From the other hand, the vector potential is determined by the current flowing through the wire.

\[A(\vec{r}, i) = \frac{\mu_0}{4\pi} \int \frac{I(t)}{r} dl \]

\[= \frac{\mu_0}{4\pi} \left[ \phi_{\text{r}} \cos(\theta - \frac{r_\perp}{c}) \frac{\omega}{c} dz \right. \right] \Rightarrow (4.12)\]

Using approximation 1 and 2

\[r_\perp \approx r(1 - \frac{d}{2r} \cos \theta) \quad \text{and} \quad \frac{1}{r_\perp} \approx \frac{1}{r} (1 + \frac{d}{2r} \cos \theta).\]

\[\sin(\omega(t - \frac{r_\perp}{c})) = \sin(\omega(t - r/c)) + \frac{\omega d}{2c} \cos \theta\]

\[\approx \sin(\omega(t - r/c)) \cos(\frac{\omega d}{2c} \cos \theta) + \cos(\omega(t - r/c)) \sin(\frac{\omega d}{2c} \cos \theta)\]

Hence, to the first order the retarded vector potential is
with \( \hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \).

Therefore,

\[
\vec{E}(\vec{r}, \theta, t) = -\nabla V - \frac{\partial \vec{A}}{\partial t}
\]

\[
\vec{E}(\vec{r}, \theta, t) = -\frac{p_0 \omega^2}{4 \pi \varepsilon_0 c^2} \frac{\cos \theta}{r} \cos[\omega(t-r/c)]\hat{r} + \frac{\mu_0 p_0 \omega^2}{4 \pi} \frac{\cos \theta}{r} \cos[\omega(t-r/c)]\hat{r} - \frac{\mu_0 p_0 \omega^2}{4 \pi} \frac{\sin \theta}{r} \cos[\omega(t-r/c)]\hat{\theta}
\]

\[
A(\vec{r}, t) = -\frac{\mu_0}{4 \pi} \int_{r/2}^{\infty} q_0 \omega \sin[\omega(t-r/c)] \left( \frac{1}{r} - \frac{d}{2r} \cos \theta \right) dz \hat{z}
\]

\[
= -\frac{\mu_0}{4 \pi} \frac{q_0 \omega d}{r} \sin[\omega(t-r/c)] \hat{z}
\]

\[
A(\vec{r}, t) = -\frac{\mu_0}{4 \pi} \frac{p_0 \omega}{r} \sin[\omega(t-r/c)] \hat{z}
\] (4.13)

Now we use these potentials (equations (4.11) and 4.13) to find the fields.

\[
\nabla V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta}
\]

\[
= -\frac{p_0 \omega}{4 \pi \varepsilon_0 c} \left\{ \frac{\partial}{\partial r} \left( \frac{\cos \theta}{r} \right) \sin[\omega(t-r/c)] \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \frac{\cos \theta}{r} \sin[\omega(t-r/c)] \right] \hat{\theta} \right\}
\]

\[
= -\frac{p_0 \omega}{4 \pi \varepsilon_0 c} \left\{ -\frac{\cos \theta}{r^2} \left( \sin[\omega(t-r/c)] + \frac{r \omega}{c} \cos[\omega(t-r/c)] \right) \hat{r} - \frac{\sin \theta}{r^2} \sin[\omega(t-r/c)] \hat{\theta} \right\}
\]

Applying approximation 3 allow us to drop the first and the third terms.

\[
\nabla V \approx -\frac{p_0 \omega^2}{4 \pi \varepsilon_0 c^2} \frac{\cos \theta}{r} \cos[\omega(t-r/c)] \hat{r}
\]

Similarly,

\[
\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega}{4 \pi c} \frac{\partial}{\partial t} \sin[\omega(t-r/c)] \hat{z} = -\frac{\mu_0 p_0 \omega^2}{4 \pi c} \cos[\omega(t-r/c)] \hat{z},
\]
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\[
\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}.
\]

Therefore,

\[
\vec{E}(\vec{r}, \theta, t) = -\nabla V - \frac{\partial \vec{A}}{\partial t}
\]

\[
\vec{E}(\vec{r}, \theta, t) = -\frac{p_0 \omega^2}{4\pi \varepsilon_0 c^2} \frac{\cos \theta}{r} \cos[\omega(t - r / c)] \hat{\rho} + \frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\cos \theta}{r} \cos[\omega(t - r / c)] \hat{\rho}
\]

\[
- \frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin \theta}{r} \cos[\omega(t - r / c)] \hat{\phi}
\]

Being on using the relation \(\mu_0 \varepsilon_0 = \frac{1}{c^2}\) the first two terms cancel each other and the electric field due to the retarded potential becomes

\[
\vec{E}(\vec{r}, \theta, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin \theta}{r} \cos[\omega(t - r / c)] \hat{\phi}
\]

(4.14)

To find the magnetic field \(\mathbf{B}\) we use the curl of a vector in a spherical coordinates system. That is
Equations (4.14) and (4.15) represent monochromatic waves of frequency traveling in the radiation direction at the speed of light. \( \mathbf{E} \) and \( \mathbf{B} \) are in phase, mutually perpendicular, and transverse; the ratio of their amplitude is \( \frac{E_0}{B_0} = C \). These waves are spherical, not plane, and their amplitude decreases like \( 1/r \) as they propagate. But for larger, they are plane over a small region. The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

\[
\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})
\]

\[
= \frac{\mu_0}{c} \left( \frac{P_0 \omega^2}{4 \pi} \sin \frac{\theta}{r} \cos[\omega(t - r/c)] \right)^2 \hat{\phi} \tag{4.16}
\]

The intensity of radiation is obtained by averaging the Poynting vector over a period of complete cycle:
From (4.17) we see that there is no radiation along the axis of the dipole \((z^+)\) (here \(\sin \theta = 0\)); the intensity profile takes the form of a donut, with its maximum in the equatorial plane \(\theta = \frac{\pi}{2}\).

The total power is obtained by integrating \(\langle \vec{s} \rangle\) over a sphere of radius \(r\):

\[
\langle P \rangle = \int \langle \vec{s} \rangle \cdot d\vec{a} = \left( \frac{\mu_0 p_0^2 \omega^4}{32\pi c} \right) \int \left( \frac{\sin \theta}{r} \right)^2 r^2 \sin \theta d\theta d\phi
\]

\[
\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \quad (4.18)
\]

As clearly seen from equation (4.18), the total power is independent of the radius of the sphere, as one would expect from conservation of energy.

### 4.1.3 Magnetic Dipole Radiation

Consider a wire loop of radius \(b\), in which we derive an alternating current:

\[
I(t) = I_0 \cos(\omega t) \quad (4.19)
\]

This is a model for an oscillating magnetic dipole

\[
\vec{m}(t) = \pi b^2 I(t) \hat{z} \\
= m_0 \cos(\omega t) \hat{z}, \quad (4.20)
\]

Where \(m_0 = \pi b^2 I_0\) is the maximum value of the magnetic dipole moment.
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is the maximum value of the magnetic dipole moment.

Here, we consider the uncharged loop so that the scalar potential is zero.

The vector potential is

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int I_0 \frac{\cos[\omega(t - \frac{r}{c})]}{r} d\vec{l} \quad (4.22)$$

From Fig. 4.4 $d\vec{l} = b \cos \phi \, d\phi$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I_0 \hat{b}}{4\pi} \int_0^{2\pi} \frac{\cos[\omega(t - \frac{r}{c})]}{r} \cos \phi \, d\phi'$$

(4.23)

Because of symmetry the x components of the potential vanish.

$$\epsilon = \sqrt{b^2 + r^2 - 2rb \cos \psi}, \quad \psi \text{ is the angle between } \vec{r} \text{ and } \vec{b}$$

$$\vec{r} = r \sin \theta \hat{x} + r \cos \theta \hat{z}, \quad \vec{b} = b \cos \phi \hat{x} + b \sin \phi \hat{y}$$

$$\Rightarrow rb \cos \psi = \vec{r} \cdot \vec{b} = rbsin \theta \cos \phi'$$

Hence, $\epsilon = \sqrt{b^2 + r^2 - 2rb \sin \theta \cos \phi'} \quad (4.24)$

For a perfect dipole, the loop has to be extremely small and as the result we have the following approximation.

Approximation 1: $b \ll r$
\[ \varepsilon \equiv r \left( 1 - \frac{b}{r} \sin \theta \cos \phi' \right) \]
\[ \text{and} \quad \frac{1}{\varepsilon} \equiv \frac{1}{r} \left( 1 + \frac{b}{r} \sin \theta \cos \phi' \right) \] (4.25)

\[
\cos[\omega(t - \varepsilon / c)] \approx \cos[\omega(t - r / c) + \frac{\omega b}{c} \sin \theta \cos \phi']
\]
\[
= \cos[\omega(t - r / c)] \cos(\frac{\omega b}{c} \sin \theta \cos \phi') - \sin[\omega(t - r / c)] \sin(\frac{\omega b}{c} \sin \theta \cos \phi')
\]

We also assume the size of the dipole is small compared to the wavelength radiated:

**Approximation 2:** \( b \ll \frac{c}{\omega} \)

For this condition, \( \cos(\frac{\omega b}{c} \sin \theta \cos \phi') \approx 1 \) and \( \sin(\frac{\omega b}{c} \sin \theta \cos \phi') \approx \frac{\omega b}{c} \sin \theta \cos \phi' \).

\[
\cos[\omega(t - \varepsilon / c)] \approx \cos[\omega(t - r / c)] \sin[\omega(t - r / c)] \frac{\omega b}{c} \sin \theta \cos \phi'
\] (4.26)

Substituting Eqs. (4.25) and (4.26) into (4.23) and dropping higher terms

\[ \vec{A}(\vec{r}, t) \equiv \frac{\mu_0 I_0}{4\pi} \int_{0}^{2\pi} \left[ \cos[\omega(t - r / c)] - \sin[\omega(t - r / c)] \frac{\omega b}{c} \sin \theta \cos \phi' \right] \cos \phi' d\phi' \]

Performing the integration and noting that the first term vanish as the result of integrating \( \cos \phi' \) from zero to \( 2\pi \). Hence,

\[
\vec{A}(\vec{r}, \theta, t) \equiv \frac{\mu_0 m_0}{4\pi} \frac{\sin \theta}{r} \left( \frac{1}{r} \cos[\omega(t - r / c)] - \frac{\omega}{c} \sin[\omega(t - r / c)] \right) \hat{\phi}
\] (4.27)

For static case \( \omega = 0 \), \( \vec{A}(\vec{r}, \theta, t) \equiv \frac{\mu_0 m_0}{4\pi} \frac{\sin \theta}{r^2} \hat{\phi} \)

In the radiation zone, we have the following approximation:

**Approximation 3:** \( r \gg \frac{c}{\omega} \)

The first term in \( \vec{A} \) (equation 4.27) is negligible as it is proportional to \( \frac{1}{r^2} \)
\( \vec{A}(r, \theta, t) = -\frac{\mu_0 m_0 \omega}{4\pi c} \frac{\sin \theta}{r} \sin[\omega(t - r/c)] \hat{\phi} \)  

Now we can calculate the fields from \( \vec{A} \) at large \( r \):

\[
\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c} \frac{\sin \theta}{r} \cos[\omega(t - r/c)] \hat{\phi} 
\]

And

\[
\vec{B} = \nabla \times \vec{A} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta A_r) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[ \frac{\partial A_r}{\partial \theta} - \frac{\partial A_\phi}{\partial r} \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi} 
\]

\[
\vec{E} = -\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\mu_0 m_0 \omega}{4\pi c} \frac{\sin \theta}{r} \sin[\omega(t - r/c)] \right) \hat{r} + \frac{\partial}{\partial r} \left( r \frac{\mu_0 m_0 \omega}{4\pi c} \frac{\sin \theta}{r} \sin[\omega(t - r/c)] \right) \hat{\phi} 
\]

\[
= -\frac{\mu_0 m_0 \omega}{4\pi c} \frac{\sin \theta}{r^2 \sin \theta} \sin[\omega(t - r/c)] \hat{r} - \frac{\mu_0 m_0 \omega^2}{4\pi c^2} \frac{\sin \theta}{r} \cos[\omega(t - r/c)] \hat{\phi} 
\]

Using approximation 3, we can drop the first term

\[
\vec{E} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2} \frac{\sin \theta}{r} \cos[\omega(t - r/c)] \hat{\phi} 
\]

Just as the electric dipole case, here also the field is perpendicular to each other and the ratio of

\[
\frac{E_0}{b_0} = C. 
\]

Note that when the oscillating field is electric dipole, \( \vec{E} \) points in the \( \hat{\theta} \) direction and \( \vec{B} \) points in \( \hat{\phi} \) direction. But as we have seen above, when the oscillating field is magnetic dipole \( \vec{E} \) points in \( \hat{\phi} \) direction and \( \vec{B} \) points along \( \hat{\theta} \). The energy flux for magnetic dipole is

\[
\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{\mu_0}{c} \left( \frac{m_0 \omega^2}{4\pi c} \frac{\sin \theta}{r} \cos[\omega(t - r/c)] \right)^2 \hat{r} 
\]

The intensity of radiation is the averaged value of the pointing vector. Hence,
The power radiated electrically is greater than the power radiated magnetically.

\[
\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{m_0}{p_0c}\right)^2,
\]

where \( m_0 = \pi b^2 I_0 \), \( p_0 = q_0 d \), \( I_0 = q_0\omega \). Setting \( d = \pi b \)

\[
\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{\omega b}{c}\right)^2
\]

But from approximation 2, \( \frac{\omega b}{c} \) is very small. Hence, electric dipole radiation dominates that of Magnetic dipole radiation.

4.1.4 Radiation from an arbitrary source

Now we consider arbitrary charge and current that is localized within some finite volume near the origin as shown in Fig. 4.5. The retarded scalar potential is
To find the fields and the power radiated from arbitrary source we follow the same approach that we have employed for the cases of oscillating electric dipole and magnetic dipole radiation.

Here also we assume that the field point \( \vec{r} \) is far away, in comparison to the dimensions of the source:

**approximation 1:** \( \vec{r}' \ll \vec{r} \)

Actually, \( r' \) is a variable of integration; approximation 1 means that the maximum value of \( r' \), as it ranges over the source, is much less than \( r \). Based on this assumption we have the following results: 

\[
\varepsilon \approx r \left( 1 - \frac{\hat{\vec{r}} \cdot \vec{r}'}{\varepsilon^2} \right), \quad \frac{1}{\varepsilon} \approx \left( 1 + \frac{\hat{\vec{r}} \cdot \vec{r}'}{\varepsilon^2} \right)
\]

and

\[
\rho(\vec{r}', t - \varepsilon / c) \approx \rho(\vec{r}', t - \frac{r}{c} + \frac{\hat{\vec{r}} \cdot \vec{r}'}{c})
\]  

(4.37)

Expanding \( \rho \) as a Taylor series in \( \varepsilon \) about the retarded time at the origin,

\[
t_0 \approx t - \frac{r}{c},
\]  

(4.38)

we get

\[
\rho(\vec{r}', t - \varepsilon / c) \approx \rho(\vec{r}', t_0) + \frac{\partial \rho(\vec{r}', t_0)}{\partial \varepsilon} \frac{\varepsilon}{c} + ... 
\]  

(4.39)

The next terms in the series would be

\[
\frac{1}{2} \beta^2 \left( \frac{\hat{\vec{r}} \cdot \vec{r}'}{c} \right)^2, \quad \frac{1}{3!} \beta^3 \left( \frac{\hat{\vec{r}} \cdot \vec{r}'}{c} \right)^3, ...
\]

We can neglect these terms because of the following approximation:
Approximation 2:

\[ r' \ll \frac{c}{|\vec{\rho}/\vec{\rho}'|}, \quad \frac{c}{|\vec{\rho}'/\vec{\rho}_0|^{1/2}}, \quad \frac{c}{|\vec{\rho}/\vec{\rho}_0|^{1/3}}, \quad \ldots \]  \hspace{1cm} (4.40)

For an oscillating system each of these ratios \( \frac{c}{\omega} \) is, and one recovers the previous approximation. In general it is more difficult to interpret (4.40), but as a procedural matter approximations 1 and 2 amount to keeping only the first-order term in \( r' \).

Inserting the approximation for \( \frac{1}{r} \) and equation (4.39) into Equation (4.36) with neglecting terms of higher order we get,

\[
V(\vec{r}, t) \equiv \frac{1}{4 \pi \varepsilon_0} \int \left( \rho(\vec{r}', t_0) + \frac{\rho(\vec{r}', t_0)}{c} \left( \frac{\vec{r}' \cdot \vec{r}'}{r^2} \right) + \ldots \right) \left( 1 + \frac{\vec{r} \cdot \vec{r}'}{r^2} \right) d\tau'
\]

\[
V(\vec{r}, t) \equiv \frac{1}{4 \pi \varepsilon_0} \int \left[ \rho(\vec{r}', t_0) d\tau' + \frac{\vec{r} \cdot \vec{r}'}{r} \int \rho'(\vec{r}, t_0) d\tau' + \frac{\vec{r} \cdot \vec{r}'}{r} \int \rho(\vec{r}, t_0) d\tau' \right].
\]

\[
V(\vec{r}, t) \equiv \frac{1}{4 \pi \varepsilon_0} \left[ \frac{Q}{r} + \frac{\vec{r} \cdot \vec{p}(t_0)}{r^2} + \frac{\vec{r} \cdot \vec{\dot{p}}(t_0)}{r c} \right] \]  \hspace{1cm} (4.41)

In the static limit the first two terms are the monopole and dipole contributions to the multiple expansions for scalar potential \( (V) \); the third term would be absent.

Let us now approximate the vector potential in similar manner. The vector potential is

\[
\vec{A}(\vec{r}, t) \equiv \frac{\mu_0}{4 \pi} \int \frac{\vec{J}(\vec{r}', t - \frac{\vec{r} - \vec{r}'}{c})}{r} d\tau'
\]  \hspace{1cm} (4.42)

For the obvious reason we will show soon, to the first order in \( r' \) to replace \( \vec{r} \) by \( r \) in the integrand and so that the vector potential takes the form

\[
\vec{A}(\vec{r}, t) \equiv \frac{\mu_0}{4 \pi r} \int \vec{J}(\vec{r}', t - r/c) d\tau' = \frac{\mu_0}{4 \pi r} \int \vec{J}(\vec{r}', t_0) d\tau' \]  \hspace{1cm} (4.43)

But we know that the dipole moment

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The field \( \vec{\mathbf{E}} \) coming from the first term in (4.41), does not contribute to the electromagnetic radiation. The radiation comes entirely from those terms in which we differentiate the argument \( t_0 \). From the defining equation (4.38) the gradient of \( t_0 \) is

\[
\nabla t_0 = -\frac{\nabla r}{c} = -\frac{1}{c} \hat{r},
\]

and hence
\[ \mathbf{\nabla} \mathbf{V} \equiv \mathbf{\nabla} \left[ \frac{1}{4\pi \varepsilon_0} \frac{\mathbf{\hat{r}} \cdot \mathbf{\hat{p}}(t_0)}{r \ c} \right] \equiv \frac{1}{4\pi \varepsilon_0} \left[ \frac{\mathbf{\hat{r}} \cdot \mathbf{\hat{p}}(t_0)}{r \ c} \right] \mathbf{\nabla} t_0 = -\frac{1}{4\pi \varepsilon_0 \ c^2} \left( \frac{\mathbf{\hat{r}} \cdot \mathbf{\hat{p}}(t_0)}{r} \right) \mathbf{\hat{r}} \]

\[ \mathbf{\nabla} \mathbf{V} \equiv -\frac{1}{4\pi \varepsilon_0 \ c^2} \left( \frac{\mathbf{\hat{r}} \cdot \mathbf{\hat{p}}(t_0)}{r} \right) \mathbf{\hat{r}} = -\frac{\mu_0}{4\pi r} [\mathbf{\hat{r}} \cdot \mathbf{\hat{p}}(t_0)] \mathbf{\hat{r}}. \]

Similar procedure can be followed to obtain the approximated expression for curl of vector potential

\[ \mathbf{\nabla} \times \mathbf{A} \equiv \frac{\mu_0}{4\pi r} \left[ \mathbf{\nabla} \times \mathbf{\hat{p}}(t_0) \right] = \frac{\mu_0}{4\pi r} \left[ \mathbf{\hat{r}} \left( \frac{\partial}{\partial t_0} \mathbf{\hat{t}}_0 \times \frac{\partial \mathbf{\hat{p}}}{\partial r} \right) \times \mathbf{\hat{p}} \right] = \frac{\mu_0}{4\pi r} \left[ \mathbf{\hat{r}} \frac{\partial \mathbf{\hat{t}}_0}{\partial r} \times \frac{\partial \mathbf{\hat{p}}}{\partial t_0} \right] \mathbf{\hat{p}}(t_0) \]

\[ \mathbf{\nabla} \times \mathbf{A} \equiv -\frac{\mu_0}{4\pi \ v} \mathbf{\nabla} t_0 \times \mathbf{\hat{p}}(t_0) \equiv -\frac{\mu_0}{4\pi r c} \mathbf{\hat{r}} \times \mathbf{\hat{p}}(t_0). \]

From the other hand the time derivative of the vector potential is

\[ \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0}{4\pi r} \mathbf{\hat{p}}(t_0). \]

With this the electric field becomes

\[ \mathbf{E}(\mathbf{r}, t) = -\mathbf{\nabla} \mathbf{V} - \frac{\partial \mathbf{A}}{\partial t} \equiv \frac{\mu_0}{4\pi r} \left[ (\mathbf{\hat{r}} \cdot \mathbf{\hat{p}}) \mathbf{\hat{r}} - \mathbf{\hat{p}} \right] = \frac{\mu_0}{4\pi r} \left[ \mathbf{\hat{r}} \times (\mathbf{\hat{r}} \times \mathbf{\hat{p}}) \right], \quad (4.46) \]

where \( \mathbf{\hat{p}} \) is evaluated at time \( t_0 = t - r / c \), and the magnetic field is

\[ \mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \equiv \frac{\mu_0}{4\pi r} \mathbf{\nabla} t_0 \times \mathbf{\hat{p}}(t_0) \equiv -\frac{\mu_0}{4\pi r c} \mathbf{\hat{r}} \times \mathbf{\hat{p}}. \quad (4.47) \]

In particular if we use spherical polar coordinates, with the z-axis is in the direction of \( \mathbf{\hat{p}}(t_0) \),

then

\[ \mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi r} \left[ \mathbf{\hat{r}} \times (\mathbf{\hat{r}} \times \mathbf{\hat{p}}) \right] \equiv \frac{\mu_0}{4\pi r} \left[ \mathbf{\hat{p}}(t_0) \right], \quad \mathbf{\hat{r}} \left( \mathbf{\hat{r}} \cdot \mathbf{\hat{z}} \right) - \mathbf{\hat{z}} \left( \mathbf{\hat{r}} \cdot \mathbf{\hat{r}} \right). \]

But we know that \( \mathbf{\hat{z}} = \cos \theta \mathbf{\hat{r}} - \sin \theta \mathbf{\hat{\theta}} \).
\[ \vec{E}(\vec{r}, t) \simeq \frac{\mu_0 \vec{p}(t_0)}{4\pi} \left( \frac{\sin \theta}{r} \right) \hat{\theta} \quad \text{and} \quad \vec{B}(\vec{r}, t) \simeq \frac{\mu_0 \vec{p}(t_0)}{4\pi c} \left( \frac{\sin \theta}{r} \right) \hat{\phi} \]  

This leads to

\[ \vec{S} \simeq \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{\mu_0 [\vec{p}(t_0)]^2}{16\pi^2 c} \left( \frac{\sin \theta}{r} \right)^2 (\hat{\theta} \times \hat{\phi} = \hat{\rho}). \]  

The Poynting vector is

\[ P \simeq \int \vec{S} \cdot d\vec{a} = \frac{\mu_0 \vec{p}^2}{6\pi c} \]  

And the total radiated power is

\[ \bar{E} \text{ and } \bar{B} \text{ are mutually perpendicular and transverse to the propagation direction } \hat{\rho}. \text{ The ratio } \frac{E_0}{B_0} = C, \text{ as always for radiation fields.} \]

**Example 1:**

An oscillating (i.e. harmonically varying) electric dipole has time-dependent dipole moment:

\[ p(t) = p_0 \cos(\omega t), \quad \text{where } \vec{p}(t) = p(t) \hat{z} = p_0 \cos(\omega t) \hat{z} \]

\[ \dot{p}(t) = -\omega p_0 \sin(\omega t) \]

\[ \ddot{p}(t) = -\omega^2 p_0 \cos(\omega t) \]  

Then we want to recover the result obtained in section (4.1.2) for potential and field. To the first order in \( r' \) the approximated scalar potential given by (4.41) is
For a single point charge $q$, the dipole moment is

$$\vec{p}(t) = q \vec{a}(t),$$

where $\vec{a}$ is the position of $q$ with respect to the origin. Accordingly,

$$\ddot{\vec{p}}(t) = q \ddot{\vec{a}}(t),$$

where $\ddot{\vec{a}}$ is the acceleration of the charge. In this case calculate the power radiated by moving charge $q$.

Everything goes through as before – get the same retarded scalar and vector potentials, Retarded $\vec{E}$ and $\vec{B}$ fields, $\tilde{S}$, $P$, etc.

In particular, the radiated electromagnetic power associated with a moving point charge $q$ is:

$$P \equiv \frac{\mu_0 \ddot{\vec{p}}^2}{6 \pi c} = \frac{\mu_0 q^2 a^2}{6 \pi c}.$$ This is the Larmour formula and we will discuss it in next section.

Note that the $EM$ power radiated by a point charge $q$ is proportional to the square of the acceleration $a$ and also to the square of the electric charge $q$. This is the origin of statement:
“Whenever one accelerates an electric charge \( q \), it radiates away EM energy in the form of (real) photons”. It is the electric dipole term which dominates this radiation process.

This is also true for *decelerating* charged particles—the *time-reversed* situation!

\( P_q \sim a^2 \) doesn’t care about sign of \( \ddot{a} \) {The EM interaction is time-reversal invariant!}

Radiation from accelerated / decelerated \( +q \) vs. \( -q \) charges is the same if \(|+q|=|-q|\).

\( P_q \) doesn’t care about the sign of \( q \)!

But: \( P_q \sim q^2 \) — so if double \( q \) — then \( P_q \) increases by factor of 4 times.

This means that for the *same* acceleration/deceleration, high-Z nuclei radiate EM energy {in the form of photons} much more than e.g. a proton (= hydrogen nucleus) — process is known as bremsstrahlung {= “braking radiation”, auf Deutsch}.

E.g. Uranium \((Z u = 92)\) gives \( 92^2 = 8464 \times \) more EM radiation than a proton for the *same* acceleration, \( a \).

4.2 Point Charge

4.2.1 Power radiated by point charge

In chapter 3 we have calculated the fields of a point charge \( q \) in arbitrary motion that was given by equations (3.53) and (3.55). That is the retarded electric field of an electric charge \( q \) in arbitrary motion is

\[
E(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{e}}{(\vec{e} \cdot \vec{u})^3} \left[ (c^2 - \nu^2)\vec{u} + \vec{e} \times (\vec{u} \times \vec{a}) \right]
\]  \hspace{1cm} (4.51)

where \( \vec{u} = c\hat{e} - \vec{v}(t_r) \), \( \vec{e} = \vec{r} - \vec{r}' = \vec{r} - \vec{u}(t_r) \), \( \nu = |\vec{r} - \vec{r}'| = c \Delta t = c(t - t_r) \)

The associated magnetic field is

\[
\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{e} \times \vec{E}(\vec{r}, t).
\]  \hspace{1cm} (4.52)

As mentioned earlier, the first term in \( \vec{E}(\vec{r}, t) \), \( \frac{q}{4\pi\varepsilon_0} \frac{\vec{e}}{(\vec{e} \cdot \vec{u})^3} (c^2 - \nu^2)\vec{u} \) is known as the generalized Coulomb field or velocity field.
The second term in $\vec{E}(\vec{r}, t)$, $\frac{q}{4 \pi \varepsilon_0} \frac{\vec{e}}{\left(\vec{e} \cdot \vec{u}\right)^2} \vec{e} \times (\vec{u} \times \vec{a})$ is known as the radiation field or acceleration field.

The retarded Poynting vector is

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\mu_0 c} [\vec{E} \times (\vec{e} \times \vec{E})] = \frac{1}{\mu_0 c} [E^2 \vec{e} - (\vec{e} \cdot \vec{E})\vec{E}].$$  \hspace{1cm} (4.53)

However, note that not all of this EM energy flux constitutes EM radiation (real photons) – some of it is still in the form of virtual photons, $\vec{S}(\vec{r}, t) = \vec{S}^{\text{vis}}(\vec{r}, t) + \vec{S}^{\text{rad}}(\vec{r}, t)$.

The radiated energy is the stuff that, in effect, detaches itself from the charge and propagates off to infinity. (It's like flies breeding on a garbage truck: Some of them hover around the truck as it makes its rounds; others fly away and never come back.) To calculate the total power radiated by the particle at time $t_r$, we draw a huge sphere of radius $r$ (Fig. 4.6), centered at the position of the particle (at time $t_r$), wait the appropriate interval

$$t - t_r = \varepsilon / c$$  \hspace{1cm} (4.54)

For the radiation to reach the sphere, and at the moment integrating the Poynting vector over the surface.

Note that the retarded time $r t$ is the correct retarded time for all points on the surface of the sphere $S'$. Again, since the area of the sphere is proportional to $r^2$, then any term in $\vec{S}$ that varies as $\frac{1}{r^2}$ will yield a finite answer for radiated EM power.
However, note that terms in $\vec{S}$ that vary as $1/\varepsilon^3$, $1/\varepsilon^4$, $1/\varepsilon^5$, etc. will contribute nothing to $\mathbf{P}$ in the limit $\varepsilon \to \infty$.

For this reason, only the acceleration fields represent true EM radiation (real photons) — (hence their other name, that of radiation fields):

$$\vec{E}_{\text{rad}} = -\frac{q}{4\pi \varepsilon_0} \frac{\varepsilon}{(\varepsilon \cdot \vec{u})^3} [\varepsilon \times (\vec{u} \times \vec{a})]$$  \hspace{1cm} (4.55)

The EM velocity fields do indeed carry EM energy — as the charged particle moves through space-time, this EM energy is dragged along with it — but it is not in the form of EM radiation.

(it is like the flies that stay with the garbage truck.) Note that, $E_{\text{rad}}(\vec{r}, t)$ is perpendicular to $\vec{\mathbf{r}}$ (due to $\vec{r} \times (\vec{u} \times \vec{a})$ term). The second term in $\vec{S}$ (Eq. 4.53) vanishes:

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c} E_{\text{rad}}^2 \hat{\varepsilon}$$  \hspace{1cm} (4.56)

Now if the point charge $q$ happened to be at rest (instantaneously at rest) at the retarded time $t_r$, then, and

$$\vec{u} = c \hat{\mathbf{r}}$$

$$\vec{E}_{\text{rad}} = -\frac{q}{4\pi \varepsilon_0 c^2} [\varepsilon \times (\varepsilon \times \vec{a})] = \frac{\mu_0 q}{4\pi} [(\varepsilon \cdot \vec{a}) \hat{\varepsilon} - \vec{a}]$$  \hspace{1cm} (4.57)

Then in this case:

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c} \left( \frac{\mu_0 q}{4\pi} \right)^2 [a^2 - (\varepsilon \cdot \vec{a})^2] \hat{\varepsilon} = \frac{\mu_0 q^2 a^2}{16\pi^2} \frac{1 - \cos^2 \theta}{\varepsilon^2} \hat{\varepsilon}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \left( \frac{\sin^2 \theta}{\varepsilon^2} \right) \hat{\varepsilon}$$  \hspace{1cm} (4.58)
where $\theta$ is the angle between $\hat{\mathbf{e}}$ and $\mathbf{a}$. No power is radiated in the forward or backward direction ($\theta = 0$ and $\theta = \pi$). Radiated power is maximum when $\theta = \pi / 2$, that is when $\hat{\mathbf{e}}$ is perpendicular to $\mathbf{a}$. In this case it is emitted in a donut about the direction of instantaneous acceleration.

The power radiated by this point charge (which is instantaneously at rest at time $t_r$) is:

$$P_{\text{rad}}(t) = \oint S_{\text{rad}}(\mathbf{r}, t) \cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2(t_r)}{16\pi^2 c} \int \frac{\sin^2 \theta}{r^2} r^2 \sin \theta d\theta d\phi$$

$$= \frac{\mu_0 q^2 a^2(t_r)}{6\pi c} \Leftarrow \text{Lamour power formula} \quad (4.59)$$